ICAF-IMDS 2020
2ND ICAN-MALAYSIA INTERNATIONAL CONFERENCE ON ACCOUNTING AND FINANCE
24-27 FEBRUARY 2020 | KUCHING, MALAYSIA, BORNEO

Available online at
http://ican-malaysia.org

2nd ICAN-Malaysia International Conference on Accounting and Finance
(ICAF-IMDS 2020)
24-27 February 2020, Kuching, Malaysia, BORNEO

Numerical Analysis of Malthus and Verhulst Growth Models using Block Method in Predicting National Population

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Abstract

Differential equation models are encountered in various fields of study such as science, technology, engineering, economics, demography, among many others. In ordinary differential equations (ODEs), population growth models are vastly adopted to investigate growth rate dynamics. This paper considers two growth models, namely the Malthus model and the Verhulst model, which describe exponential and logistic growth. These models are solved by employing a highly accurate block method developed with a higher derivative to improve the accuracy resulting from its maximal order. Convergence properties of the block method are investigated to validate its usability. They can be applied to solve the growth models expressed in the ODEs form. The method's accuracy is compared to other recently introduced numerical methods adopted for the ODE growth models' solution, such as the Chebyshev collocation method and the modified Heun’s iterative method. The solution obtained is then utilized to obtain a population prediction for Nigeria. The prediction results obtained indicate that they agree with the results available in the literature, thus confirming the proposed method's reliability and applicability.

Keywords: Ordinary differential equations, growth models, block method, national population, Nigeria

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1. INTRODUCTION

Differential equations have shown broad applicability in various fields such as engineering, biological sciences, economics, and other sciences fields. These differential equation models in which a dependent variable and one independent variable are related, known as ordinary differential equations (ODEs), have a range of modelling growth in various application areas.

The commonly adopted population growth ODE models known as the Malthus model and Verhulst model, which describe exponential and logistic growth, have been studied by Huang (2014) to obtain a population prediction of Huanggang City in China, and also Andriani, Suyitno, and Junaidi, (2019). They specifically applied the Verhulst model to estimate the population of Bandar Lampung, Indonesia. Likewise, Lal (2018) and Ekele (2018) studied this growth above models to investigate India and Nigeria's population dynamics and growth, respectively. These individual studies inform the relevance of these growth models in recent literature; however, a numerical insight is yet to be explored concerning the population growth predictions.

Studies are continuously developed to develop new numerical methods with improved accuracy compared to existing approaches for solving ordinary differential equation models. The trend has evolved from the simplest numerical methods such as Euler and Runge-Kutta methods to highly accurate block methods (Zainuddin et al.,

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2016; Ramos, Mehta, & Vigo-Aguiar, 2017; Omar & Kuboye, 2018). In the application of numerical methods to exponential and logistic population growth models, Pirzada, Shaikh, and Shah (2018) modified Heun's iterative method to solve some sample population growth rate problems, while Öztürk, Anapalı, and Gülsu (2017) adopted a numerical scheme using Chebyshev coefficients to solve the logistic population growth model. However, the methods were not adopted to predict the national population.

Hence, this article introduces a new approximate solution to Malthus and Verhulst growth models using an efficient block method developed with a higher derivative. The next section of this article discusses the population growth models. It develops the block method to solve the two models. The convergence properties of the block method are also stated in Section 2. Section 3 validates the block method by comparing the solutions to Chebyshev and modifying Heun's methods. In contrast, Section 4 applies the block method to the population prediction of Nigeria using the growth models. Finally, Section 5 concludes this article.

2. METHODOLOGY

This section details the population growth models' solution approach using the block method and investigates its convergence properties. The exponential growth model coined the Malthus model follows the primary hypothesis that the population growth rate is constant. This hypothesis implies that the population growth in unit time is proportional to the population at that time. The model assumes that the population value $p(t)$ is a continuously differentiable function of the time t and can be expressed as

$$\frac{dp}{dt} = rp, \; p(t_0) = p_0 \tag{1}$$

where r denotes the net growth rate of population and $p(t_0)$ is an initial condition chosen as a known population value p_0 at a time t_0. The solution to the Malthus model in Equation (1) is given by

$$p = p_0 e^{-r(t_0 - t)} . \tag{2}$$

However, this model is unrealistic in a real-world setting because it implies that the population can grow without bounds as time goes on. It is also known that various factors may affect and limit the growth rate of a particular population, such as birth rate, death rate, food supply, and predators. The constant growth r usually considers the birth and death rates but none other factors. It can be interpreted as a net (birth minus death) percent growth rate per unit time. Thus, whether the population growth rate stays constant or changes over time comes to light. Scientists have found that in many biological systems, the population grows until a specific steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that only a certain number can thrive without running into resource issues in a given area.

An organism's carrying capacity in a given environment refers to be the maximum population of that organism that the environment can sustain indefinitely. Denoting the carrying capacity by K, and still have r been an actual number that represents the growth rate. The function $p(t)$ represents the population value as a function of time t, and the constant p_0 represents the initial population $t = 0$. Then the Verhulst population model is defined as

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \; p(t_0) = p_0 \tag{3}$$

The solution to the Verhulst model in Equation (3) can be obtained by

$$p = \frac{Kp_0 e^{rt}}{Ke^{t_0 r} - p_0 e^{t_0 r} + p_0 e^{rt}} . \tag{4}$$

To develop the block method to solve (1) and (3), maximal order $\vartheta = 2(k+1)$ is ensured by choosing the steplength value $k = 2$. This guarantees improved accuracy over the conventional low accuracy one-step methods. The algorithm developed by Adeyeye and Omar (2018) is adopted to obtain the expressions for the block method as:

$$\begin{aligned} p_{n+1} &= p_n + \frac{101hp_n^{(1)}}{240} + \frac{8hp_{n+1}^{(1)}}{15} + \frac{11hp_{n+2}^{(1)}}{240} + \frac{13h^2 p_n^{(2)}}{240} - \frac{h^2 p_{n+1}^{(2)}}{6} - \frac{h^2 p_{n+2}^{(2)}}{80} \\ p_{n+2} &= p_n + \frac{7hp_n^{(1)}}{15} + \frac{16hp_{n+1}^{(1)}}{15} + \frac{7hp_{n+2}^{(1)}}{15} + \frac{h^2 p_n^{(2)}}{15} - \frac{h^2 p_{n+2}^{(2)}}{15} \end{aligned} \quad (5)$$

Thus, the block method schemes are combined as simultaneous integrators for the solution of the growth models.

Investigating the block method's convergence properties in (5), the block method should satisfy zero-stability and consistency to ensure convergence. Hence, to examine its zero-stability, the block method is normalized to give the first characteristic polynomial as:

$$\rho(R) = \det\left(R_{ij}A^0 - A^1\right) = R(R-1), \quad (6)$$

where A^0 is the identity matrix of order 2 and $A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Since the roots of $\rho(R) = 0$ satisfying $|R_j| \le 1,\ j = 1, 2$, then the block method is zero stable (Butcher, 2008).

In particular, the block method is consistent if it has order $\vartheta \ge 1$. To investigate the order of the block method, the linear operator $L_h[p(t)]$ is defined as:

$$L_h[p(t)] = \begin{bmatrix} p_{n+1} - \left(\sum_{j=0}^{k-2} \alpha_{1j} p_{n+j} + \sum_{i=0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} p_{n+j}^{(i)} \right) \\ p_{n+2} - \left(\sum_{j=0}^{k-2} \alpha_{2j} p_{n+j} + \sum_{i=0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} p_{n+j}^{(i)} \right) \end{bmatrix} \quad (7)$$

Equation (7) can be rewritten as:

$$L_h[p(t)] = \begin{bmatrix} p_{n+1} - \left(\sum_{j=0}^{k-2} \alpha_{1j} p_{n+j} + \sum_{i=0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} f^{(i-1)}\left(t_n + jh, p\left(t_n + jh\right)\right) \right) \\ p_{n+2} - \left(\sum_{j=0}^{k-2} \alpha_{2j} p_{n+j} + \sum_{i=0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} f^{(i-1)}\left(t_n + jh, p\left(t_n + jh\right)\right) \right) \end{bmatrix} \quad (8)$$

which becomes an equation of the form below (after expanding $p(t_n + jh)$ and $f^{(i-1)}\left(t_n + jh, p(t_n + jh)\right)$ in a Taylor series about t_n),

$$L_h[p(t)] = \begin{bmatrix} C_{10} p(t_n) + C_{11} hp^{(1)}(t_n) + \cdots + C_{1\vartheta} h^{(\vartheta)} p^{(\vartheta)}(t_n) + \cdots \\ C_{20} p(t_n) + C_{21} hp^{(1)}(t_n) + \cdots + C_{2\vartheta} h^{(\vartheta)} p^{(\vartheta)}(t_n) + \cdots \end{bmatrix} \quad (9)$$

The block method is of order ϑ if $C_{q0} = C_{q1} = \cdots = C_{q\vartheta} = 0,\ C_{q(\vartheta+1)} \ne 0;\ q = 1, 2$, and for the block method (5) $C_{q0} = C_{q1} = \cdots = C_{q6} = 0;\ q = 1, 2$. This implies that the block method is of order $\vartheta = 6$, satisfying the maximal order criterion $\vartheta = 2(k+1)$.

Therefore, since the block method satisfies both conditions of zero-stability and consistency, it is likewise convergent.

3. VALIDATION OF THE BLOCK METHOD

This section compares the block method's solution with the Chebyshev expansion method in Pirzada et al. (2018) and the modified Heun's method in Öztürk et al. (2017).

Firstly, consider the exponential population growth rate problem

$$\frac{dp}{dt} = kp,\ t \in [0,1] \tag{10}$$

with the exact solution given by $p(t) = 100e^{0.250679566129t}$ an initial condition $p(0) = 100$ with $k = 0.250679566129$.

A comparison with the modified Heun's method is given in Table 1 below.

Table 1: Comparison of the Modified Heun's Method (MHM) and Block Method (BM)

t	**Exact Solution**	**Absolute Error (MHM)**	**Absolute Error (BM)**
0.1	102.53847998347332	0.088	1.421085e-14
0.2	105.14139877321155	0.180	0.000000e+00
0.3	107.81039213541338	0.277	1.421085e-14
0.4	110.54713735987491	0.379	1.421085e-14
0.5	113.35335431405804	0.486	1.421085e-14
0.6	116.23080652391599	0.598	1.421085e-14
0.7	119.18130228215516	0.716	1.421085e-14
0.8	122.20669578463047	0.839	1.421085e-14
0.9	125.30888829558741	0.969	1.421085e-14
1.0	128.48982934248377	1.104	2.842171e-14

Next, we consider the logistic population growth rate problem form

$$\frac{dp}{dt} = p(1-p),\ t \in [0,1] \tag{11}$$

with the exact solution is obtained to be $p(t) = \frac{-1}{e^{(-t-\ln(2))} - 1}$ an initial condition $p(0) = 2$. A comparison with the Chebyshev expansion method is given in Table 2 below.

Table 2. Comparison of the Chebyshev Expansion Method (CEM) and Block Method (BM)

t	**Exact Solution**	**Absolute Error (CEM)**	**Absolute Error (BM)**
0.1	1.8262128682421239	-	5.149214e-13
0.2	1.6930941063701719	0.880e-5	5.895284e-13
0.3	1.5883330213710647	-	5.471179e-13
0.4	1.5041213444160908	0.141e-4	4.791723e-13
0.5	1.4352665983935839	-	4.114487e-13
0.6	1.3781808411258631	0.170e-4	3.519407e-13
0.7	1.3303049418392214	-	3.015366e-13
0.8	1.2897642077008449	0.212e-4	2.591261e-13
0.9	1.2551537079897774	-	2.240430e-13
1.0	1.2253996735605639	0.249e-3	1.940670e-13

From the results displayed in Tables 1 and 2, the accuracy of the block method is established. Note that the solutions provided by Pirzada et al. (2018) for the Chebyshev expansion method in Table 2 only presented values at 0.2, 0.4, 0.6, 0.8, and 1.0. The following section further adopts the block method to obtain a population prediction for Nigeria. A comparison is made with prediction results in the literature.

4. POPULATION PREDICTION USING GROWTH MODELS

The Malthus and Verhulst models defined in Equations (1) and (3) respectively will be adopted to predict Nigeria's population. A comparison will be made with the exact solution of the ODE models and other prediction results already obtained in the literature. The data culled from World Bank Group (2020) selected the population statistics data of Nigeria from 1983 to 2013 (in hundred thousand). They then used the Malthus and Verhulst models to predict the population value for subsequent years till 2050. Tables 3 and 4 show the solution obtained using the block method in column 3, compared to the exact solution of the ODE models in column 2 and the prediction by the World Bank Group in column 1. The initial conditions were chosen with corresponding values of the 2013 data, and all predictions are in bold fonts.

Table 3: Prediction Results using the Exact Solution of the Malthus Model and the Block Method

Year	Population Value	Malthus Model -Exact Solution	Block Method -Malthus Model
2013	1717.65769	1717.65769	1717.65769
2014	1764.04902	**1762.37870**	**1762.37870**
2015	1811.37448	**1808.26407**	**1808.26407**
2016	1859.60289	**1855.34411**	**1855.34411**
2017	1908.73311	**1903.64992**	**1903.64992**
2018	1958.74740	**1953.21343**	**1953.21343**
2019	**2009.64000**	**2004.06738**	**2004.06738**
2020	**2061.40000**	**2056.24536**	**2056.24536**
2021	**2114.01000**	**2109.78186**	**2109.78186**
2022	**2167.47000**	**2164.71223**	**2164.71223**
2023	**2221.82000**	**2221.07277**	**2221.07277**
2024	**2277.13000**	**2278.90071**	**2278.90071**
2025	**2333.43000**	**2338.23427**	**2338.23427**
2030	**2629.77000**	**2658.89450**	**2658.89450**
2035	**2949.86000**	**3023.52935**	**3023.52935**
2040	**3290.67000**	**3438.16942**	**3438.16942**
2045	**3647.12000**	**3909.67231**	**3909.67231**
2050	**4013.15000**	**4445.83604**	**4445.83604**

To obtain a Verhulst model solution, the carrying capacity value, as obtained by Gabriel (2018) as 8891134631, was utilized to obtain the solution in the table below.

Table 4: Prediction Results using the Exact Solution of the Verhulst Model and the Block Method

Year	Population Value	Verhulst Model -Exact Solution	Block Method -Verhulst Model
2013	1717.65769	1717.65769	1717.65769
2014	1764.04902	**1762.37869**	**1762.37869**
2015	1811.37448	**1808.26405**	**1808.26405**
2016	1859.60289	**1855.34408**	**1855.34408**
2017	1908.73311	**1903.64988**	**1903.64988**
2018	1958.74740	**1953.21338**	**1953.21338**
2019	**2009.64000**	**2004.06732**	**2004.06732**
2020	**2061.40000**	**2056.24529**	**2056.24529**
2021	**2114.01000**	**2109.78176**	**2109.78176**
2022	**2167.47000**	**2164.71212**	**2164.71212**
2023	**2221.82000**	**2221.07264**	**2221.07264**
2024	**2277.13000**	**2278.90057**	**2278.90057**
2025	**2333.43000**	**2338.23410**	**2338.23410**
2030	**2629.77000**	**2658.89421**	**2658.89421**
2035	**2949.86000**	**3023.52891**	**3023.52891**
2040	**3290.67000**	**3438.16876**	**3438.16876**
2045	**3647.12000**	**3909.67134**	**3909.67134**
2050	**4013.15000**	**4445.83468**	**4445.83468**

The results obtained in Tables 3 and 4 have shown the block method efficiently, obtaining the same values as the exact solutions in both growth models. Another comparison considered a similar study by Gabriel (2018). The author adopted the exponential growth model to predict Nigeria's population to be 456103747 in 2050, while the solution by the author using the logistic growth model predicted the population of Nigeria to be 452614778 in 2050. The predictions by the block method to solve the growth models are in closer agreement to the values proposed by the World Bank Group (2020) in comparison to the predictions by Gabriel (2018).

5. CONCLUSION

This article has shown the adoption of block methods to solve ordinary differential equations modelling concepts in demography. The developed block method satisfied its convergence properties and further displayed superior accuracy than the Chebyshev expansion method and modified Heun's method, and both were adopted to solve growth models. The method was adopted to solve the growth models and compare them with their exact solutions. The methods efficiently obtained accurate solutions as the growth models. Future research aims to explore growth models with consideration of relevant additional variables. It is expected that a better prediction could be obtained with the consideration of other parameters in developing the growth models. However, the possibility of these additional variables making it difficult to obtain an exact solution is likely. However, the block method developed in this article has shown that even in the absence of exact solutions, there is a guarantee of a highly accurate solution.

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